February 12, 1999

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PROBLEM 2358. In triangle ABC, let the mid-points of BC, CA, AB be L, M, N respectively, and let the feet of the altitudes from A, B, C be D, E, F, respectively. Let X be the intersection of LE and MD, let Y be the intersection of MF and NE, and let Z be the intersection of ND and LF. Show that X, Y, Z are collinear.

SOLUTION. Let $ND \cap EL = P$, $MF \cap EL = Q$ and $MF \cap ND = R$. Using Menelaus's theorem on the triangle PQR with the points {F, Z, L}, {M, D, X} and {E, N, Y} respectively we have

$$\frac{QF}{FR}\frac{RZ}{ZP}\frac{PL}{LQ}=1,\quad \frac{QM}{MR}\frac{RD}{DP}\frac{PX}{XQ}=1,\quad \frac{QY}{YR}\frac{RN}{NP}\frac{PE}{EQ}=1.$$

We multiply this three equalities and get

$$\frac{QF}{FR}\frac{RZ}{ZP}\frac{PL}{LQ}\frac{QM}{MR}\frac{RD}{DP}\frac{PX}{XQ}\frac{QY}{YR}\frac{RN}{NP}\frac{PE}{EQ}=1. \qquad (1)$$

The points N, F, D, L, E, M lie on the Euler's circle for triangle ABC (see "Geometry revisited", H. Coxeter, S. Greitzer – theorem 1.81). Hence

$$QE \,.\, QL = QM \,.\, QF \,;\; RN \,.\, RD = RF \,.\, RM \,;\; PL \,.\, PE = PD \,.\, PN.$$

From (1) it follows that

$$\frac{RZ}{ZP}\frac{PX}{XQ}\frac{QY}{YR}=1.$$

Using again Menelaus's theorem we prove that X, Y and Z are collinear.
This problem is known Paskal's theorem. See "Geometry revisited", H. Coxeter, S. Greitzer – theorem 3.81.