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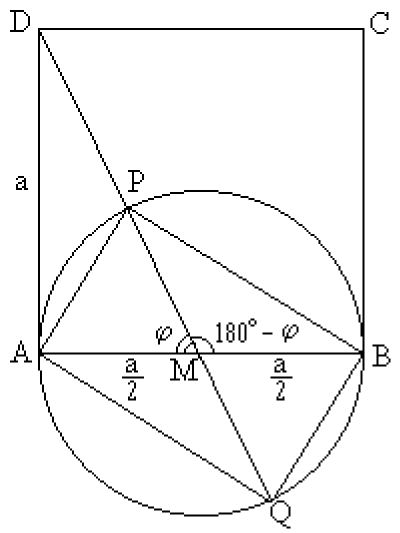
Problem 2813. Suppose that **M** is the mid-point of side **AB** of the square **ABCD**. Let **P** and **Q** be the points of intersection of the line **MD** with the circle, centre **M**, radius **MA** (=**MB**), where **P** is inside the square **ABCD** and **Q** is outside. Prove that rectangle **APBQ** is a golden rectangle; that is **PB** : **PA** = $\left(\sqrt{5}+1\right)$: 2.

Solution:



Let $\angle$**AMP** = φ and **AB** = **a** ⇒ $\angle$**BMP** = 180^0 - φ, **AM** = **MB** = $\frac{\mathbf{a}}{2}$. We use the Pythagoras theorem for Δ**AMD** and get

$$\mathbf{DM}=\sqrt{\mathbf{AM}^2+\mathbf{AD}^2}=\sqrt{\left(\frac{\mathbf{a}}{2}\right)^2+\mathbf{a}^2}=\frac{\mathbf{a}\sqrt{5}}{2}.$$

From Δ**AMD** ⇒ $\cos\varphi=\frac{\mathbf{AM}}{\mathbf{DM}}=\frac{1}{\sqrt{5}}$.

We use the Law of Cosines for Δ**AMP** and Δ**BMP** and find:

$$\mathbf{AP}^2=\mathbf{AM}^2+\mathbf{MP}^2-2.\mathbf{AM}.\mathbf{MP}.\cos\varphi=$$

$$=\frac{\mathbf{a}^2}{4}+\frac{\mathbf{a}^2}{4}-2.\frac{\mathbf{a}}{2}.\frac{\mathbf{a}}{2}.\frac{1}{\sqrt{5}}=\frac{\mathbf{a}^2}{2}.\frac{\sqrt{5}-1}{\sqrt{5}}; \qquad (1)$$

$$\mathbf{BP}^2=\mathbf{BM}^2+\mathbf{PM}^2-2.\mathbf{BM}.\mathbf{MP}.\cos(180^0-\varphi)=\frac{\mathbf{a}^2}{4}+\frac{\mathbf{a}^2}{4}+2.\frac{\mathbf{a}}{2}.\frac{\mathbf{a}}{2}.\frac{1}{\sqrt{5}}=\frac{\mathbf{a}^2}{2}.\frac{\sqrt{5}+1}{\sqrt{5}}. \qquad (2)$$

From (1) and (2) ⇒ $\frac{\mathbf{BP}}{\mathbf{AP}}=\sqrt{\frac{\mathbf{BP}^2}{\mathbf{AP}^2}}=\sqrt{\frac{\sqrt{5}+1}{\sqrt{5}-1}}=\sqrt{\frac{\left(\sqrt{5}+1\right)^2}{4}}=\frac{\sqrt{5}+1}{2}$.