September 19, 1999

Mitko Hristov Kunchev
BABA TONKA High School of Mathematics
18 Ivan Vazov St.
ROUSSE 7000, BULGARIA
direktor@mg-babatonka.bg

BRUCE SHAWYER
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland
CANADA. A1C 5S7

Problem 2447. Two circles intersect at P and Q. A variable line through P meets the circles again at A and B. Find the locus of the orthocentre of triangle ABQ.

Solution. Let the circles be $K_1(O_1; R_1)$ and $K_2(O_2; R_2)$. We denote by AA_1, BB_1, QQ_1 the altitudes of the triangle ABQ, and by H – its orthocentre. Let $AH \cap K_1 = D$ and $BH \cap K_2 = C$ (Figure 1). Let $\angle BAQ = \alpha$ and $\angle ABQ = \beta$. It is clear that α and β are constant – they don't depend on the position of line AB. We denote by φ the angle between K_1 and K_2. It's clear that $\varphi = \alpha + \beta$.

We split the problem into three cases.

Case I. Let $\varphi = \alpha + \beta = 90^0$. Now ΔABQ is a right triangle and Q is its orthocentre. Hence the locus contains only one point – Q. We have this case when the angle between the two given circles is 90^0.

Case II. Let $0^0 < \varphi = \alpha + \beta < 90^0$. Then $\angle BAQ$ and $\angle ABQ$ are acute angles and $\angle AQB$ is obtuse angle.

Case II. 1. Let $\angle PAQ = \alpha$, i. e. the point A lies on the bigger arc PQ and $A \neq P$, $A \neq Q$ (Figure 1).

Triangle ABA_1 is a right triangle, so $\angle BAA_1 = 90^0 - \beta$. The angle BAA_1 is an inscribed angle in K_1, then $\overset{\frown}{PD} = 2.\angle BAA_1 = 2.(90^0 - \beta) \rightarrow$ constant.

Hence D doesn't depend on the line AB, i. e. the altitude AA_1 goes always trough the fixed point D. Analogously in triangle ABB_1 $\angle ABB_1 = 90^0 - \alpha$. It is inscribed angle in K_2, so $\overset{\frown}{PC} = 2.\angle ABB_1 = 2.(90^0 - \alpha) \rightarrow$ constant. Hence C doesn't depend on the line AB, i. e. the altitude BB_1 goes always trough the fixed point C. (The third altitude QQ_1 goes always trough the fixed point Q, i. e. the three altitudes of the triangle ABQ go trough three fixed points)

We'll prove that $O_1 \in QC$.

ΔPQO_1 is an isosceles triangle ($O_1P = O_1Q = R_1$) and $\angle PO_1Q = \overset{\frown}{PQ} = 2.\alpha$, then

$$\angle PQO_1 = \frac{1}{2}.\left(180^0 - 2\alpha\right) = 90^0 - \alpha .$$

Quadrilateral $PBCQ$ is inscribed in K_2, hence

$$\angle PQC = 180^0 - \angle PBC = 180^0 - (90^0 - \alpha) = 90^0 + \alpha ;$$

$$\angle PQO_1 + \angle PQC = 90^0 - \alpha + 90^0 + \alpha. = 180^0.$$

Thus $O_1 \in QC$. Analogously $O_2 \in DQ$. This is very important! We discovered that $C = O_1Q \cap K_2$ and $D = O_2Q \cap K_2$!

Quadrilateral $APQD$ is inscribed in K_1, so $\angle APQ + \angle ADQ = 180^0$, but $\angle ADQ + \angle QDH = 180^0$, then $\angle APQ = \angle QDH$.

Quadrilateral $BCQP$ is inscribed in K_2, so $\angle BCQ + \angle BPQ = 180^0$, but $\angle BCQ + \angle QCH = 180^0$, then $\angle BPQ = \angle QCH$.

Since $\angle APQ + \angle BPQ = 180^0$ it follows that $\angle QDH + \angle QCH = 180^0$. Hence the points D, Q, C, H are concyclic, i. e. the point H lies always on the circumscribed circle of ΔDQC (It doesn't depend on the position of line AB).

Case II. 2. Let $\angle PAQ = 180^0 - \alpha$, i. e. the point A lies on the smaller arc PQ and $A \neq P$, $A \neq Q$ (Figure 2).

Triangle ABB_1 is a right triangle, so

$$\angle ABB_1 = 90^0 - \alpha \text{ and } \angle PBC = 180^0 - \angle ABB_1 = 90^0 + \alpha;\ \overset{\frown}{PC}{}^* = 2(90^0 + \alpha).$$

Triangle BAA_1 is a right triangle, so

$$\angle BAA_1 = 90^0 - \beta \text{ and } \angle PAD = 180^0 - \angle BAA_1 = 90^0 + \beta;\ \overset{\frown}{PD}{}^* = 2(90^0 + \beta).$$

Hence the points D and C don't depend on the line AB. We point that:

$$\overset{\frown}{PC}{}^* + \overset{\frown}{PC} = 2\left(90^0 + \alpha + 90^0 - \alpha\right) = 360^0,$$

$$\overset{\frown}{PD}{}^* + \overset{\frown}{PD} = 2\left(90^0 + \beta + 90^0 - \beta\right) = 360^0.$$

Hence the points D and C from II. 1. and II. 2. are identical!

Again we'll prove that the points D, Q, C, H are concyclic.

$$\angle DQC = \angle O_1QO_2 = \angle O_1QP + \angle O_2QP = 90^0 - \alpha + 90^0 - \beta = 180^0 - \alpha - \beta. \qquad (1)$$

ΔHQ_1B is a right triangle, so $\angle Q_1BH = \angle ABB_1 = 90^0 - \alpha \Rightarrow \angle Q_1HB = \alpha$.

ΔHQ_1A is also a right triangle, so $\angle Q_1AH = \angle BAA_1 = 90^0 - \beta \Rightarrow \angle Q_1HA = \beta$.

We get:

$$\angle AHC = \angle Q_1HA + \angle Q_1HB = \alpha + \beta \Rightarrow \angle DHC = 180^0 - \angle AHC = 180^0 - \alpha - \beta. \qquad (2)$$

From (1) and (2) immediately follows that $\angle DQC = \angle DHC$. Hence the points D, Q, C, H are concyclic, i. e. the point H lies always on the circumscribed circle of ΔDQC (It doesn't depend on the line AB).

Case II. 3. If $A \equiv P$ or $A \equiv Q$ hence ΔABQ doesn't exist.

We conclude that the orthocentre of the triangle ABQ always lies on the circumscribed circle of the triangle DQC.

Let H is an arbitrary point on the circumscribed circle of the triangle DQC, $DH \cap K_1 = A$ and $CH \cap K_2 = B$. We'll prove that $P \in AB$.

The points D, Q, C, H are concyclic, so $\angle QDH + \angle QCH = 180^0$. (3)

The points A, P, Q, D are concyclic, so $\angle ADQ = 180^0 - \angle APQ$, but $\angle ADQ = 180^0 - \angle QDH$, then $\angle APQ = \angle QDH$. (4)

The points P, B, C, Q, are concyclic, so $\angle BCQ = 180^0 - \angle BPQ$, but $\angle BCQ = 180^0 - \angle QCH$, then $\angle BPQ = \angle QCH$. (5)

From (3), (4) and (5) it follows that $\angle APQ + \angle BPQ = 180^0 \Rightarrow P \in AB$.

We proved that the locus of the orthocentre of triangle ABQ is the circumscribed circle of the triangle DQC, without the point Q, where Q is a given point, $D = O_2Q \cap K_1$ and $C = O_1Q \cap K_2$.

Case III. Let $90^0 < \varphi = \alpha + \beta < 180^0$. This case is the same as Case II.