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# TWO TRIGONOMETRICAL PROBLEMS IN A TRIANGLE

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**Problem 1.** The angles of a triangle $\alpha$ and $\beta$, $\alpha \le 90^0$, $\beta \le 90^0$ satisfy the equation:

$$\frac{1}{2}+\frac{1}{2}\cos^2(\alpha-\beta)=\sin 2\alpha+\sin 2\beta-\cos(\alpha-\beta).$$

Determine the type of this triangle.

**Solution:** Since the expression is symmetric to $\alpha$ and $\beta$ let's assume that $\alpha \ge \beta$. Transform the equation in the following way:

$$1+\cos^2(\alpha-\beta)=4\sin(\alpha+\beta)\cos(\alpha-\beta)-2\cos(\alpha-\beta);$$

$$[1+\cos(\beta-\beta)]^2=4\sin(\alpha+\beta)\cos(\alpha-\beta).$$

As $0 \le \alpha-\beta < 90^0$ and $0<\alpha+\beta<180^0$, follows $\cos(\alpha-\beta)>0$ and $\sin(\alpha+\beta)>0$. Then we get

$$1+\cos(\alpha-\beta)=2\sqrt{\sin(\alpha+\beta)}\sqrt{\cos(\alpha-\beta)};$$

$$\frac{1}{\sqrt{\cos(\alpha-\beta)}}+\sqrt{\cos(\alpha-\beta)}=2\sqrt{\sin(\alpha+\beta)}.$$

As $\frac{1}{\sqrt{\cos(\alpha-\beta)}}+\sqrt{\cos(\alpha-\beta)} \ge 2$ and $2\sqrt{\sin(\alpha+\beta)} \le 2$, follows that $\sin(\alpha+\beta)=1$ and $\cos(\alpha-\beta)=1$. We get $\alpha=\beta$ and $\alpha+\beta=90^0$, i.e. the triangle is isosceles and rectangular.

**Problem 2.** The angles $\alpha, \beta, \gamma$ of a triangle satisfy the inequality:

$$\sqrt{3}\cos\gamma+3\sin\alpha\cos\beta-4\sin^2\alpha \ge 4\cos^2\alpha+\cos\alpha\sin\beta.$$

Find the angles of the triangle.

**Solution:** As $\gamma=180^0-(\alpha+\beta)$, we transform the inequality as follows

$$-\sqrt{3}\cos(\alpha+\beta)+\sin\alpha\cos\beta+\cos\alpha\sin\beta \ge 4(\sin^2\alpha+\cos^2\alpha)-2\sin\alpha\cos\beta+2\cos\alpha si\beta$$

$$-\sqrt{3}\cos(\alpha+\beta)+\sin(\alpha+\beta) \ge 4-2\sin(\alpha-\beta)$$

$$-\frac{\sqrt{3}}{2}\cos(\alpha+\beta)+\frac{1}{2}\sin(\alpha+\beta) \ge 2-\sin(\alpha-\beta)$$

$$\cos 150^0\cos(\alpha+\beta)+\sin 150^0\sin(\alpha+\beta) \ge 2-\sin(\alpha-\beta)$$

$$\cos[150^0-(\alpha+\beta)] \ge 2-\sin(\alpha-\beta)$$

As $\cos[150^0-(\alpha+\beta)] \le 1$ follows that $1 \ge 2-\sin(\alpha-\beta)$, i.e. $\sin(\alpha-\beta) \ge 1$. We get that $\sin(\alpha-\beta)=1$ and then $\cos[150^0-(\alpha+\beta)]=1$. For the angles we get $\alpha-\beta=90^0$ and $\alpha+\beta=150^0$. From the last equalities follows that $\alpha=120^0, \beta=\gamma=30^0$.