September 19, 1999
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Problem 2441. Suppose that $D$, $E$, $F$ are the mid-points of the sides $BC$, $CA$, $AB$ of $\Delta ABC$. The incircle of $\Delta AEF$ touches $EF$ at $X$, the incircle of $\Delta BFD$ touches $FD$ at $Y$, and the incircle of $\Delta CDE$ touches $DE$ at $Z$. Show that $DX$, $EY$, $FZ$ are collinear. What is the intersection point?

Solution 2441. Let $AB = c$, $BC = a$, $AC = b$. Then

$$AE = EC = FD = \frac{b}{2};\ AF = FB = ED = \frac{c}{2};\ BD = DC = EF = \frac{a}{2}.$$

The incircles of the triangles $AEF$, $BFD$, $CDE$ touch $EF$, $FD$, $ED$ at $X$, $Y$, $Z$ respectively. Hence

$$FY = P_{FBD} - BD = \frac{-a+b+c}{4};\ YD = P_{FBD} - FB = \frac{a+b-c}{4};$$
$$DZ = P_{EDC} - EC = \frac{a+c-b}{4};\ ZE = P_{EDC} - CD = \frac{b+c-a}{4}; \qquad (1)$$
$$EX = P_{AFE} - AF = \frac{a+b-c}{4};\ XF = P_{AFE} - AE = \frac{a+c-b}{4}.$$

We calculate an expression

$$\frac{FY}{YD}\frac{DZ}{ZE}\frac{EX}{XF} = \frac{-a+b+c}{a+b-c}\frac{a+c-b}{b+c-a}\frac{a+b-c}{a+c-b} = 1.$$

According to Ceva's theorem we get that $DX$, $EY$, $FZ$ are collinear.
From (1) it follows that the excircles of $\Delta FDE$ touch its sides at the points $X$, $Y$, $Z$, i. e. the intersection point $N$ is the point of Nagel for $\Delta FDE$.