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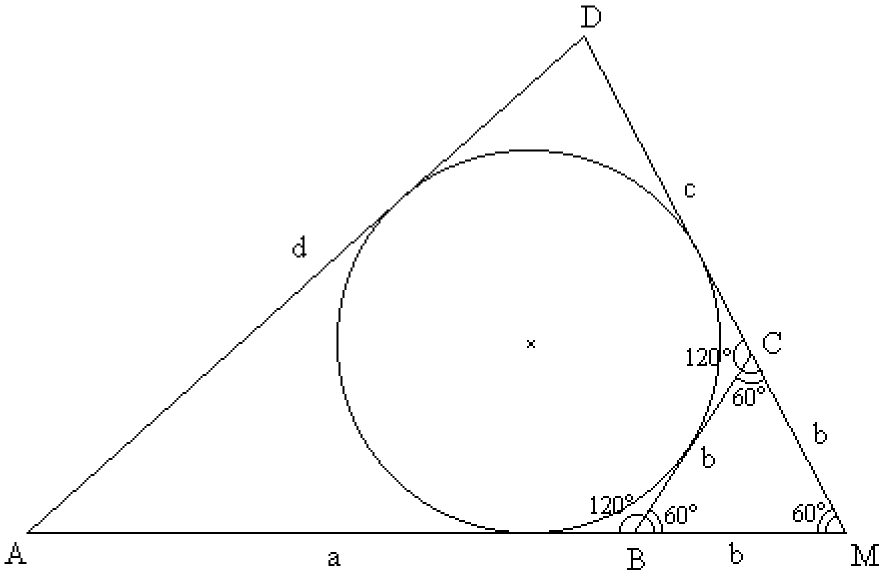
Problem 2849. In a convex quadrilateral **ABCD**, we have $\angle ABC = \angle BCD = 120^0$. Suppose that

$$AB^2 + BC^2 + CD^2 = AD^2.$$

Prove that **ABCD** has an inscribed circle.

Solution :



Let $AB = a$, $BC = b$, $CD = c$, $AD = d$. The given equality is $d^2 = a^2 + b^2 + c^2$. (1)
The quadrilateral **ABCD** has an inscribed circle if and only if **AB** + **CD** = **BC** +**AD**. We will prove that $a + c = b + d$.

$\angle ABC = \angle BCD = 120^0 \Rightarrow$ the lines **AB** and **CD** are not parallel. We denote their intersection point by **M**. $\angle MBC = 180^0 - \angle ABC = 60^0$ and $\angle MCB = 180^0 - \angle BCD = 60^0$, because they are adjacent angels. Hence ΔBMC is equilateral, $\angle BMC = 60^0$ and $BM = MC = BC = b$. So we have that $AM = a + b$ and $DM = c + b$.

We split the problem in two cases.

Case I. Let $a = c$. Then $AM = a + b = c + b = DM$. Therefore ΔAMD is equilateral, because $\angle AMD = 60^0$. We get $AD = AM = DM$, i.e. $d = a + b \Rightarrow d^2 = a^2 + 2ab + b^2$. (2)

From (1) we have that $d^2 = a^2 + c^2 + b^2 = 2a^2 + b^2$. (3)

From (2) and (3) comes $a^2 + 2ab + b^2 = 2a^2 + b^2 \Rightarrow a = 2b$ and $d = a + b = 3b$. We got that $AB + CD = BC + AD = 4b$ and them the problem is solved.

Case II. Let $\mathbf{a} \neq \mathbf{c}$. Without loss of generality, we may assume that $\mathbf{a} > \mathbf{c}$. We use the Law of Cosines for $\Delta\mathbf{AMD}$: $\mathbf{AD}^2 = \mathbf{AM}^2 + \mathbf{DM}^2 - 2\mathbf{AM}.\mathbf{DM}.\cos 60^0 \Leftrightarrow$

$$\Leftrightarrow \mathbf{d}^2 = (\mathbf{a}+\mathbf{b})^2 + (\mathbf{b}+\mathbf{c})^2 - 2.(\mathbf{a}+\mathbf{b}).(\mathbf{c}+\mathbf{b})\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \mathbf{d}^2 = \mathbf{a}^2 + 2\mathbf{ab} + \mathbf{b}^2 + \mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2 - \mathbf{ac} - \mathbf{ab} - \mathbf{bc} - \mathbf{b}^2 \Leftrightarrow$$

$$\Leftrightarrow \mathbf{d}^2 = \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{ab} + \mathbf{bc} - \mathbf{ac} \qquad (4)$$

From (1) and (4) $\Rightarrow$ $\mathbf{ab} + \mathbf{bc} - \mathbf{ac} = 0 \Rightarrow 2\mathbf{ac} - 2\mathbf{ab} - 2\mathbf{bc} = 0$. (5)

From (1) and (5) $\Rightarrow$ $\mathbf{d}^2 = \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 - 2\mathbf{ab} - 2\mathbf{bc} + 2\mathbf{ac} \Leftrightarrow \mathbf{d}^2 = (\mathbf{a} + \mathbf{c} - \mathbf{b})^2 \Leftrightarrow$

$$\Leftrightarrow \mathbf{d} = |\mathbf{a}+\mathbf{c}-\mathbf{b}| \Leftrightarrow \mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b} \text{ or } \mathbf{d} = -\mathbf{a} - \mathbf{c} + \mathbf{b}.$$

If $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$ then $\mathbf{d} + \mathbf{b} = \mathbf{a} + \mathbf{c}$ and them the problem is solved.

Let $\mathbf{d} = -\mathbf{a} - \mathbf{c} + \mathbf{b}$. We know that $\mathbf{a} > \mathbf{c} \Rightarrow \mathbf{AM} > \mathbf{DM} \Rightarrow \angle\mathbf{ADM} > \angle\mathbf{DAM}$, but $\angle\mathbf{AMD} = 60^0 \Rightarrow \angle\mathbf{DAM} < 60^0 \Rightarrow 60^0 > \angle\mathbf{DAM} > \angle\mathbf{CAM}$. In $\Delta\mathbf{AMC}$ we have $\angle\mathbf{CAM} < \angle\mathbf{AMC} \Rightarrow \mathbf{AC} > \mathbf{MC}$, but $\mathbf{AD} + \mathbf{DC} > \mathbf{AC}$, hence $\mathbf{AD} + \mathbf{DC} > \mathbf{MC}$, i.e. $\mathbf{d} + \mathbf{c} > \mathbf{b}$. We have that $\mathbf{d} = -\mathbf{a} - \mathbf{c} + \mathbf{b} \Rightarrow$
$-\mathbf{a} - \mathbf{c} + \mathbf{b} + \mathbf{c} > \mathbf{b} \Rightarrow \mathbf{a} < 0$. So this case is not possible.

Hence in the quadrilateral **ABCD** can be inscribed circle.