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Problem 2462. If the angles A, B, C of ΔABC satisfy $\cos A \sin\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2}$, prove that ΔABC is isosceles.

SOLUTION **A**

$$\cos A \sin\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \frac{1}{2}\sin\frac{3A}{2} - \frac{1}{2}\sin\frac{A}{2} = \frac{1}{2}\cos\frac{B-C}{2} - \frac{1}{2}\cos\frac{B+C}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\sin\frac{3A}{2} - \frac{1}{2}\sin\frac{A}{2} = \frac{1}{2}\cos\frac{B-C}{2} - \frac{1}{2}\cos\frac{180^0 - A}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\sin\frac{3A}{2} - \frac{1}{2}\sin\frac{A}{2} = \frac{1}{2}\sin\left(90^0 - \frac{B-C}{2}\right) - \frac{1}{2}\sin\frac{A}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\sin\frac{3A}{2} - \frac{1}{2}\sin\left(90^0 - \frac{B-C}{2}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sin\frac{A-C}{2}\cos\frac{A-B+180^0}{2} = 0 \Leftrightarrow A = C \vee A = B \quad \left(A, B, C \in \left(0^0, 180^0\right)\right).$$

We proved that the triangle is isosceles.

SOLUTION **B**

Using the Law of cosines and Euler's theorem we get

$$\cos A \sin\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{b^2 + c^2 - a^2}{2bc} \cdot \sqrt{\frac{(p-b)(p-c)}{bc}} = \sqrt{\frac{(p-c)(p-a)}{ac}} \cdot \sqrt{\frac{(p-a)(p-b)}{ab}} \Leftrightarrow$$

$$\frac{b^2 + c^2 - a^2}{2bc} = \frac{p-a}{a} \Leftrightarrow ab^2 + ac^2 - a^3 = cb^2 + bc^2 - abc \Leftrightarrow$$

$$\Leftrightarrow b^2(a-c) - a\left(a^2 - c^2\right) + bc(a-c) = 0 \Leftrightarrow (a-c)\left(b^2 - a^2 - ac + bc\right) = 0 \Leftrightarrow$$ We

$$\Leftrightarrow (a-c)(b-a)(a+b+c) = 0 \Leftrightarrow a = c \vee a = b.$$

proved that the triangle is isosceles.